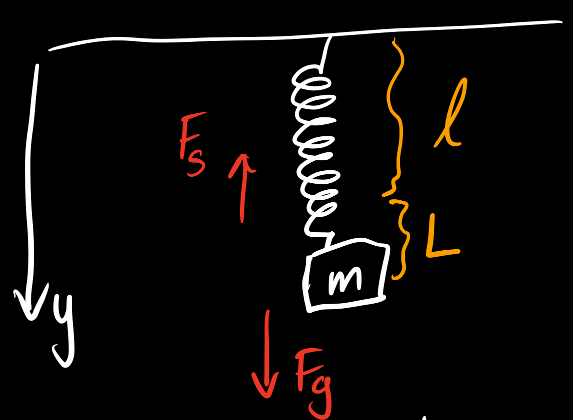
§3.7 Mechanical and Electrical Vibrations

Many physical problems can be described by the IVP

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_0'$$

How to interpret a, b, and c?

Mass on a Spring



- mass m hanging from spring
- spring has unstretched length l

Gravitational force $F_g = mg$

- g is the gravitational acceleration

Spring restoring force $F_s = -KL$ (Hooke's law)

- K is the spring constant
- L is the displacement from spring rest length l

The mass is in equilibrium, so

$$F_g + F_s = 0$$

$$mg - KL = 0.$$

For dynamics, we want to study the motion of the mass if it is initially displaced or is acted on by an external force.

Let $u(t)$ be the displacement from equilibrium. Then Newton's law of motion says

$$m u'' = f(t)$$

Where $f(t)$ is the net force acting on the mass.

Forces to consider:

1. Gravity : $F_g = mg$
2. Spring force : $F_s = -K(L+u)$
3. Damping (air resistance, energy dissipation, friction, mechanical)
   - Acts in direction opposite motion
   - Assume viscous damping
     - resistive force is proportional to speed of the mass

$$F_d = -\gamma u'$$

- γ is the damping constant

4. Applied Force $F(t)$

Now, we have

$$mu'' = mg - k(L+u) - \gamma u'(t) + F(t)$$

$$mu'' + \gamma u' + ku = mg - kL + F(t)$$

or

$$mu'' + \gamma u' + ku = F(t)$$

since $mg - kL = 0$.

Finally, we need to specify an initial position and initial velocity of the mass

$$u(0) = u_0 \quad , \quad u'(0) = v_0 \qquad .$$

Ex A mass weighing 4 lbs stretches a spring 2 inches. Suppose that the mass is given an additional 6 inch positive displacement in positive direction and then released. The mass exerts a viscous resistance of 6 lbs when mass has a velocity of 3 ft/s.

Set up the initial value problem.

Let u be the displacement from equilibrium.

First, the mass is

$$m = \frac{4 \text{ lb}}{32 \frac{\text{ft}}{\text{s}^2}} = \frac{1}{8} \text{ slug } \left(\frac{\text{lb s}^2}{\text{ft}} \right)$$

The damping constant γ is

$$F_d = -\gamma u'$$

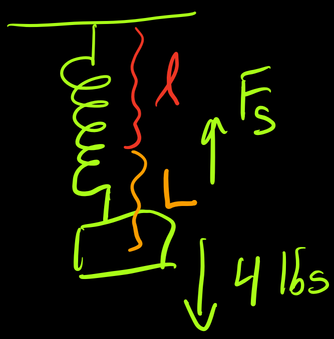
$$+6 = +\gamma \cdot 3$$

$$\gamma = \frac{6 \text{ lb}}{3 \text{ ft/s}} = 2 \ \frac{\text{lb}\cdot\text{s}}{\text{ft}}$$

The spring constant K is

$$F_s = -KL$$

$$-4 = F_s = -KL$$



$$-4 = -K \cdot 2$$

$$K = \frac{4 \text{ lb}}{2 \text{ in}} \cdot \frac{12 \text{ in}}{1 \text{ ft}} = 24 \ \frac{\text{lb}}{\text{ft}}$$

Thus, $m = \frac{1}{8}$ slug, $\gamma = 2 \ \frac{\text{lb}\cdot\text{s}}{\text{ft}}$, and $K = 24 \ \frac{\text{lb}}{\text{ft}}$.

therefore, our DE is

$$\frac{1}{8}u'' + 2u' + 24u = 0$$

and ICs are

$$u(0) = \frac{1}{2} \text{ ft.}, \quad u'(0) = 0.$$

What are possible solution behaviors?
(without external forcing)

$$mu'' + \gamma u' + ku = 0$$

The characteristic equation is

$$mr^2 + \gamma r + k = 0$$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Solution has 3 forms (real part of roots < 0)

$\gamma^2 - 4mk > 0$

$$u = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad \text{(over damped)}$$

$\gamma^2 - 4mk = 0$

$$u = (c_1 + c_2 t) e^{-\frac{\gamma t}{2m}} \quad \text{(critically damped)}$$

$\gamma^2 - 4mk < 0$

$$u = e^{-\frac{\gamma t}{2m}} \left(c_1 \cos(\mu t) + c_2 \sin(\mu t) \right)$$

where $\mu = \dfrac{\sqrt{4km - \gamma^2}}{2m}$

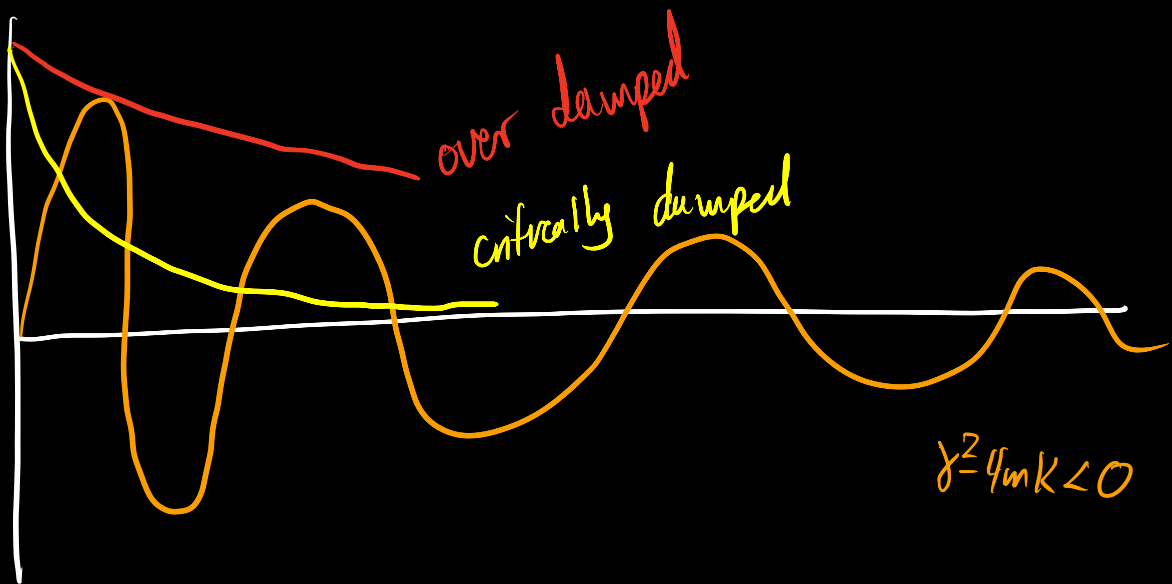
<u>Note:</u> When $\gamma^2 - 4mk < 0$ and if $\gamma = 0$, we get pure oscillations, simple harmonic motion.

- μ is the natural frequency
- Period is $T = \dfrac{2\pi}{\mu}$
- Maximum displacement is the amplitude of the motion.

<u>Note 2</u>: When $\gamma \neq 0$

- μ is called the quasi-frequency

• T is called the quasi-period



Additional application: Electrical circuits

§3.8 Forced Periodic Vibrations

We will now consider an external force $F(t)$,

$$mu'' + \gamma u' + ku = F(t)$$

Suppose $F(t) = F_0 \cos(\omega t)$, i.e. a periodic force of amplitude F_0 and frequency ω. Then

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

and we expect the solution to be of the form

$$\begin{aligned} u &= u_c + u_p \\ &= c_1 u_1 + c_2 u_2 + A\cos(\omega t) + B\sin(\omega t) \end{aligned}$$

We know $u_c \to 0$ as $t \to \infty$ since $m, \gamma, k > 0$.

- u_c is called the transient solution
- u_p persists indefinitely and is called the steady-state solution or the forced response of the system

* The effect of the ICs is lost in the presence of damping

Let's take a closer look at

$$u_p = A\cos(\omega t) + B\sin(\omega t).$$

After some manipulations, we may write

$$u_p = R\cos(\omega t - \delta)$$

where

$$R = \frac{F_0}{\Delta}, \quad \cos\delta = \frac{m(\omega_0^2-\omega^2)}{\Delta}$$

and $$\sin\delta = \frac{\gamma\omega}{\Delta}$$

where

$$\Delta = \sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

and $$\omega_0^2 = \frac{k}{m}$$

"natural frequency"

$F_0 = KL$

$\frac{F_0}{K} = L$

Now,

$$\frac{R}{F_0/K} = \frac{K}{\Delta}$$

$$= \left(\left(1 - \left(\frac{w}{w_0}\right)^2\right)^2 + \beta \left(\frac{w}{w_0}\right)^2 \right)^{-\frac{1}{2}}$$

where $\beta = \frac{\gamma^2}{mK}$

- $\frac{R}{F_0/K}$ is the ratio of the force response amplitude to displacement caused by force F_0.

$w \to 0$ (low frequency limit)

$$\frac{R}{F_0/k} \longrightarrow 1 \quad \text{or} \quad R \to \frac{F_0}{k}$$

$\underline{\omega \to \infty}$ (high frequency limit)

$$R \to 0$$

There is an intermediate value of ω where R has a maximum

$$\omega_{max}^2 = \omega_0^2 \left(1 - \frac{\gamma^2}{2mk}\right)$$

$$R_{max} = \frac{F_0}{\gamma \omega_0 \sqrt{1 - \left(\frac{\gamma^2}{4mk}\right)}}$$

For small γ

$$R_{max} \simeq \frac{F_0}{\gamma \omega_0}$$

and as γ gets smaller, R_{max} grows

This is called Resonance.

Other areas to explore

- Forced vibrations without damping
  - beats
  - amplitude modulation